J. DIFFERENTIAL GEOMETRY
12 (1977) 565-566

ON A THEOREM OF AL'BER ON SPACES OF MAPS

VAGN LUNDSGAARD HANSEN

In [1, Theorem 7] (see also [2, Theorem 32]) Al'ber proved

Theorem 1 (*Al'ber*). *Let V be a compact connected Riemannian manifold, and let M be a compact connected Riemannian manifold with strictly negative sectional curvature. Let $C^2(V, M)$ denote the space of maps of class C^2 of V into M equipped with the C^2-topology. Then one of the following holds for any (path-) component K in $C^2(V, M)$:*

(1) *K has the homotopy type of a point, and contains a unique harmonic map,*

(2) *K has the homotopy type of a circle, and all the harmonic maps in K map V with the same value of the Dirichlet integral into the same closed geodesic of M,*

(3) *K has the homotopy type of M, and each harmonic map in K maps V into a single point of M.*

Theorem 1 can also be proved by the methods developped by Eells and Sampson [3] and is very close to being stated explicitely in Hartman [6].

The purpose of this note is to point out that the topological aspect of Theorem 1 is a simple consequence of classical knowledge about the fundamental group of a Riemannian manifold with negative sectional curvature and the following elementary result in homotopy theory.

Lemma. *Let X be a locally compact connected CW-complex, and Y a space of type $(\pi, 1)$. Let $C(X, Y)$ denote the space of continuous maps of X into Y equipped with the compact-open topology. For any based map $f: X \to Y$ denote by $C(X, Y; f)$ the (path-) component in $C(X, Y)$ containing f, and denote by $C(\pi; f)$ the centralizer of $f_*(\pi_1(X))$ in $\pi_1(Y)$. Then $C(X, Y; f)$ is a space of type $(C(\pi; f), 1)$.*

We recall that a connected CW-complex Y is called a space of type $(\pi, 1)$, if π is a group, $\pi_i(Y) = 0$ for $i \geq 2$, and $\pi_1(Y) \simeq \pi$. We recall also that if A is a subset of the group G, then the centralizer of A in G is the subgroup $C_A(G) = \{g \in G \mid ag = ga, \text{ all } a \in A\}$.

A proof of the lemma can be found in Gottlieb [4, Lemma 2].

It is a classical result of Hadamard-Cartan that a complete Riemannian manifold M with nonpositive sectional curvature is a space of type $(\pi_1(M), 1)$. From the classical results of Preissmann [8] it also follows that the funda-

Communicated by J. Eells, Jr., October 25, 1975.

mental group of a compact manifold M with strictly negative sectional curvature has

Property C. A group π is said to have property C if any centralizer in π is either the identity subgroup, or an infinite cyclic group, or π.

The necessary Riemannian geometry to prove the results above concerning the fundamental group of a Riemannian manifold with negative sectional curvature can also be found in Gromoll, Klingenberg and Meyer [5, § 7.2].

The following theorem contains the topological aspect of the theorem of Al'ber.

Theorem 2. *Let X be a locally compact connected CW-complex, and Y a space of type $(\pi, 1)$ where π has property C. Then any component in $C(X, Y)$ has the homotopy type of either a point, or a circle, or Y.*

Theorem 2 follows by observing that the component determined by the based map $f: X \to Y$ is a space of type $(C(\pi; f), 1)$ and that $C(\pi; f)$ is either the identity subgroup, or an infinite cyclic group, or $\pi_1(Y)$.

We should also remark that the space of maps $C^2(V, M)$ has the same homotopy type as $C(V, M)$ by well-known approximation theorems or, for an elegant proof, by Palais [7, Theorem 13. 14].

Al'ber's proof of Theorem 1 involves fairly advanced calculus of variations, and it is necessary for his proof that the domain is compact. So apart from a completely topological setting we obtain also in Theorem 2 a slight generalization of the topological part of Theorem 1 in so far that we only need the domain to be locally compact. It would be interesting to have an example of a manifold M, which is a space of type $(\pi, 1)$ where π has property C, but where M does not admit a Riemannian metric with strictly negative sectional curvature.

References

[1] S. I. Al'ber, *Spaces of mappings into a manifold with negative curvature,* Dokl. Akad. Nauk SSSR **178** (1968) 13–16; Soviet Math. Dokl. **9** (1968) 6–9.

[2] ——, *The topology of functional manifolds and the calculus of variations in the large,* Russian Math. Surveys **25** (1970) 51–117.

[3] J. Eells & J. H. Sampson, *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. **86** (1964) 109–160.

[4] D. H. Gottlieb, *Covering transformations and universal fibrations,* Illinois J. Math. **13** (1969) 432–437.

[5] D. Gromoll, W. Klingenberg & W. Meyer, *Riemannsche Geometrie im Grossen,* Lecture Notes in Math. Vol. 55, Springer, Berlin, 1968.

[6] P. Hartman, *On homotopic harmonic maps,* Canad. J. Math. **19** (1967) 673–687.

[7] R. S. Palais, *Foundations of global non-linear analysis,* Math. Lecture Note Series, Benjamin, New York, 1968.

[8] A. Preissmann, *Quelques propriétés globales des espaces de Riemann,* Comment. Math. Helv. **15** (1943) 175–216.

University of Copenhagen